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LETTER TO THE EDITOR

Fluid formulation of a generalised Schrödinger–Langevin equation

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Abstract. We recast a generalised Schrödinger-Langevin equation into a set of hydrodynamical equations and draw some analogies with a classical (plasma) fluid theory describing the motion of charged particles in a neutralised background.

In recent years there has been renewed interest in the analogy between the quantum mechanics of a point particle and the dynamics of a fluid (Guerra 1981, Ghosh and Deb 1982, 1984, Putterman and Roberts 1983, Nonnemacher *et al* 1983, Takabayasi 1983, Bohm and Hiley 1984, Nassar 1984). Madelung was the first to transform the Schrödinger equation for a particle into two fluid-dynamical equations: a continuity equation and an Euler-type equation. This description involves the density ρ $(\equiv\psi^*\psi)$ and the velocity field v as primary quantities. Thus, the fluid dynamicist can gather experience of its effects by translating some of the elementary situations of the quantum theory into their corresponding fluid mechanical statements and vice versa.

In fact, some years ago Nelson (1966) showed that a natural and straightforward particle interpretation of the Madelung fluid is indeed possible by allowing a random character to the underlying trajectories: a stochastic interpretation of quantum mechanics in terms of subquantum random fluctuations resulting from the action of a stochastic invariant thermostat (see Guerra 1981, for a detailed review of this theory). By proceeding in the same line of reasoning, we have recently derived a generalised Schrödinger-Langevin equation (GNLSLE) for the description of an interacting, non-conservative system. Thus, by taking the ensemble average of this equation we recovered the celebrated Langevin equation (Nassar 1985).

The aim of this letter is to Madelung-transform the GNLSLE and show its striking similarities with the classical (plasma) fluid theory describing the motion of charged particles (electrons) is a neutralised (ion) background subject to an external electric field, namely

$$\partial\rho(x, t)/\partial t+\nabla\cdot(\rho(x, t)v(x, t))=0 \tag{1}$$

and

$$\frac{\partial v(x, t)}{\partial t}+v(x, t)\cdot\nabla v(x, t)+\nu v(x, t)+\frac{kT}{m}\frac{\nabla\rho(x, t)}{\rho(x, t)}=-\frac{eE(x, t)}{m}, \tag{2}$$

where m, $-e$, ρ, v, $(kT/m)^{1/2}$, ν are the particle (electron) mass, charge, density, mean velocity, thermal speed and collision frequency, respectively (Krall and Trivelpiece 1973, Tanenbaum 1967, Mengoli *et al* 1983).

Therefore, let us recall the GNLSLE (Nassar 1985, Enz 1979)

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(x,t)+\left(\frac{\hbar\nu}{2i}\ln\frac{\psi(x,t)}{\psi^*(x,t)}+b\ln|\psi(x,t)|^2+ex\cdot E(x,t)\right)\psi(x,t) \qquad (3)$$

where $\psi(x, t)$ is the wavefunction, $E(x, t)$ is the external electric field, $(\hbar\nu/2i)\ln(\psi/\psi^*)$ is the Kostin quantum 'potential' (Kostin 1972, Yasue 1978) and $b\ln|\psi|^2$ is the Bialynicki-Birula–Mycielski quantum 'potential' (Bialynicki-Birula and Mycielski 1976).

To obtain the fluid dynamical description of the wavefunction $\psi(x, t)$, we express this function in the Madelung form

$$\psi(x,t)=\phi(x,t)\exp(iS(x,t)). \qquad (4)$$

After substitution of (4) into (3), we obtain, from its imaginary and real parts, respectively

$$\partial\rho(x,t)/\partial t+\nabla\cdot(\rho(x,t)v(x,t))=0 \qquad (5)$$

and

$$\frac{\partial v(x,t)}{\partial t}+v(x,t)\cdot\nabla v(x,t)+\nu v(x,t)+\frac{b}{m}\frac{\nabla\rho(x,t)}{\rho(x,t)}=-\frac{eE(x,t)}{m}-\nabla\left(\frac{V_{qu}}{m}\right) \qquad (6)$$

where $\rho(x, t)=|\phi(x, t)|^2$ is the quantum fluid-particle density, $v(x, t)=(\hbar/m)\nabla S(x, t)$ is the fluid-particle velocity, and $V_{qu}=-(\hbar^2/2m)(\rho(x, t))^{-1/2}\nabla^2(\rho(x, t))^{1/2}$ is the Bohm quantum potential.

Despite the striking mathematical similarities between (6) and (2), physically they need to be considered carefully. The quantum fluid (6), under an ensemble average, corresponds exactly to the classical Langevin equation

$$\ddot{x}+\nu\dot{x}=-eE(x,t)/m \qquad (7)$$

whereas the classical fluid (2) is similar to (7) except for the nonlinear term $v\cdot\nabla v$ (which is called the inertial term in hydrodynamics) and the pressure term $p=(kT/m)\rho$. It only attains the form of (7) if, for instance, v and its derivatives are small and we consider the cold-plasma approximation. Nevertheless, we still can gather some important consequent analogies between (6) and (2), which simplify our physical interpretation of the GNLSLE and render strong support for exploration of manifold solutions, based on the well developed mathematical problem solving methods of nonlinear differential equations in the literature.

Previously, the term $\mu\equiv V_{qu}/m=-(\hbar^2/2m^2)(\rho)^{-1/2}\nabla^2(\rho)^{1/2}$ had been loosely referred to as a quantum pressure term. Later it was realised (Grant 1973, Roberts 1976, Wong 1976) that this is a misnomer because (a) the dimensionality is incorrect and it would be better regarded as a quantum (chemical) potential per unit mass (b) the word pressure suggests a phenomenon that depends only on the local thermodynamic state (here fixed ρ) and (c) the presence of derivatives in

$$\nabla_i\mu=-(\nabla_j\sigma_{ij})/\rho, \qquad (8)$$

where

$$\sigma_{ij}=(\hbar^2\rho/4m^2)\nabla_i\nabla_j\ln\rho \qquad (9)$$

shows that neighbouring points are involved in its definition. Thus, σ_{ij} is a properly dimensioned contender for the (anisotropic) quantum stress tensor, instead.

The right contender in (6) for the quantum (barotropic) pressure is the term

$$(b/m)\rho \qquad (10)$$

where b plays a somewhat similar role to kT in (2). The key difference between the quantum pressure and the classical pressure is that (10) (as well as (9)) is an intrinsic, self-information content intimately attached to the stochastic invariant thermostat. If $b>0$ ($b<0$) we have repulsive (attractive) type of interactions, in accordance with Bialynicki-Birula and Mycielski's findings: for a logarithmic Schrödinger-like equation with nonlinearity $b\ln|\psi|^2$ they obtained (in the case of attractive interactions $b<0$) soliton-like solutions of Gaussian form.

Finally, unlike the two other nonlinear quantum potentials, the Kostin quantum potential, viewed through (6), has a precise classical counterpart: the dissipative term νv in (2) or (7). It is a quantum transcription of a phenomenologically classical dissipative and irreversible process.

In passing, Caldeira and Leggett (1985) have very recently given a possible justification for the use of nonlinear wavefunctions for the description of non-conservative systems, based on their conclusion that damping tends to destroy interference effects of two Gaussian wavepackets in a harmonic potential. Therefore, we believe that the protocol presented above can be used as a clue to a deeper understanding of nonlinear wave mechanics as well as to encourage usage of the well developed mathematical problem solving methods of nonlinear differential equations to gather new insights and design some physical applicabilities to investigate the range of validity of our GNLSLE. We intend to do this in the future in a longer work.

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